



APRIL/MAY 2025

**GMA43/DMA43 — FUNCTIONAL
ANALYSIS**

Time : Three hours

Maximum : 75 marks

SECTION A — (10 × 2 = 20 marks)

Answer ALL questions.

18. Let H be a Hilbert space, and let f be an arbitrary functional in H^* . Then prove that there exists a unique vector y in H such that $f(x) = (x, y)$ for every x in H .
19. Prove that $r(x) = \lim_{n \rightarrow \infty} \|x^n\|^{\frac{1}{n}}$, where $r(x)$ is spectral radius.
20. If M is a maximal ideal in A , then the Banach Algebra $\frac{A}{M}$ is a division algebra, and therefore equals the Banach algebra C of complex numbers. The natural homomorphism $x \rightarrow x + M$ of A onto $\frac{A}{M} = C$ assigns to each elements x in A a complex number $x(M)$ defined by $x(M) = x + M$ and the mapping $x \rightarrow x(M)$ has the following properties :
- (a) $(x + y)(M) = x(M) + y(M)$
 - (b) $(\alpha x)(M) = \alpha x(M)$
 - (c) $(xy)(M) = x(M)y(M)$
 - (d) $x(M) = 0 \Leftrightarrow x \in M$
 - (e) $I(M) = 1$
 - (f) $|x(M)| \leq \|x\|$ – Prove.

- 1. Define normed linear space.
- 2. Define Isometric Isomorphism.
- 3. Write about projection.
- 4. Define Hilbert Space.
- 5. Define Adjoint of an operator.
- 6. Write about Normal Operator.
- 7. Define Banach algebra.
- 8. Define Topological Divisors.
- 9. Write about B^* -algebra.
- 10. Define Gelfand Mapping.

SECTION B — ($5 \times 5 = 25$ marks)

Answer ALL questions.

11. (a) If M is a closed linear subspace of a normed linear space N and x_0 is a vector not in M , then there exists a functional f_0 in N^* such that $f_0(M) = 0$ and $f_0(x_0) \neq 0$.

Or

- (b) State and prove Holder's inequality.
12. (a) State and prove Open Mapping Theorem.

Or

- (b) Prove that closed convex subset C of a Hilbert space H contains a unique vector of smallest norm.
13. (a) If N is a normal operator on H , then prove that $\|N^2\| = \|N\|^2$.

Or

- (b) If A_1 and A_2 are self-adjoint operators on H , then prove that their product $A_1 A_2$ is self-adjoint $\Leftrightarrow A_1 A_2 = A_2 A_1$.
14. (a) Every element x for which $\|x - 1\| < 1$ is regular, and the inverse of such an element is given by the formula $x^{-1} = 1 + \sum_{n=1}^{\infty} (1 - x)^n$.
Prove.

Or

- (b) Prove that $\sigma(x^n) = \sigma(x)^n$.

15. (a) If f_1 and f_2 are multiplicative functionals on A with the same null space M , then prove that $f_1 = f_2$.

Or

- (b) Prove that the maximal ideal space M is a compact Hausdorff Space.

SECTION C — ($3 \times 10 = 30$ marks)

Answer any THREE questions.

16. Let N and N' be normed linear spaces and T a linear transformation of N into N' . Then prove that the following conditions on T are all equivalent to one another.

- (a) T is continuous;
(b) T is continuous at the origin, in the sense that $x_n \rightarrow 0 \Rightarrow T(x_n) \rightarrow 0$;
(c) there exists a real number $K \geq 0$ with the property that $\|T(x)\| \leq K\|x\|$ for every $x \in N$
(d) if $S = \{x : \|x\| \leq 1\}$ is the closed unit sphere in N , then its image $T(S)$ is bounded set in N' .

17. State and prove Bessel's Inequality.

